NOTES ON HIGHER-DIMENSIONAL TARAI FUNCTIONS

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0. Introduction

I. Takeuchi defined the following recursive function, called the tarai function, in [5].

$$t(x,y,z) = \textbf{ if } x \leq y \textbf{ then } y \textbf{ else } t(t(x-1,y,z), t(y-1,z,x), t(z-1,x,y))$$

This function requires many recursive calls even for small x, y, and z, so it is used to see how effectively the programming language implementation handles recursive calls. In [3], J. McCarthy proved that this recursion terminates without call-by-need and t can be computed in the following way.

$$t(x,y,z) = \textbf{ if } x \leq y \textbf{ then } y \textbf{ else if } y \leq z \textbf{ then } z \textbf{ else } x$$

In [4], J. S. Moore gave an easier proof and verified it by the Boyer-Moore theorem prover.

D. Knuth proposed the following generalization in [2], called the n-dimensional tarai function.

$$\begin{aligned} t(x_1, x_2, \ldots, x_n) = &\textbf{ if } x_1 \leq x_2 \textbf{ then } x_2 \\ &\textbf{ else } t(t(x_1 - 1, x_2, \ldots, x_n), \ldots, t(x_n - 1, x_1, \ldots, x_{n-1})) \end{aligned}$$

It was shown by T. Bailey, J. Coldwell, and J. Cowles that the 4-dimensional tarai function does not terminate without call-by-need because

$$\begin{aligned} t(3,2,1,5) &= t(t(2,2,1,5), t(1,1,5,3), t(0,5,3,2), t(4,3,2,1)) \\ &= t(2,1,5,4) \\ &= t(t(1,1,5,4), t(0,5,4,2), t(4,4,2,1), t(3,2,1,5)) \end{aligned}$$

T. Bailey and J. Cowles announced in [1] that they gave an informal (handwritten) proof of the following conjecture.

Conjecture 1. Let $n \geq 3$ be an integer. Define the function f on $\mathbb{Z}^n$ by

$$\begin{aligned} f(x_1, x_2, \ldots, x_m) = &\textbf{ if } (\exists k < m)(x_1 > x_2 > \cdots > x_k \leq x_{k+1}) \\ &\textbf{ then } g_b(x_1, x_2, \ldots, x_{k+1}) \\ &\textbf{ else } x_1 \end{aligned}$$

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where the function g_b is defined by

$$
\begin{aligned}
g_b(x_1, x_2, \ldots, x_j) = \ & \textbf{if } j \leq 3 \textbf{ then } x_j \\
& \qquad\qquad \textbf{else} \ \ \textbf{if } x_1 = x_2 + 1 \text{ or } x_2 > x_3 + 1 \\
& \qquad\qquad\qquad\quad \textbf{then } g_b(x_2, \ldots, x_j) \\
& \qquad\qquad \textbf{else} \ \max\{x_3, x_j\}
\end{aligned}
$$

Then, f satisfies the n-dimensional tarai recurrence.

The goal of this paper is to give a proof to this theorem. Moreover, the proof will be simpler than the one proposed in [1], so we hope that it is easier to be formalized.

1. Termination with call-by-need

In this section, we shall prove that the n-dimensional tarai function is a total function for every $n \geq 3$. Throughout this section, let n be a fixed natural number with $n \geq 3$ and t the n-dimensional tarai function.

First, we shall prepare notation. Let $\vec{x} = \langle x_1, \cdots, x_n \rangle \in \mathbb{Z}^n$. Define $\sigma(\vec{x})$ and $r(\vec{x})$ by

$$
\begin{aligned}
\sigma(\vec{x}) &= \langle x_1 - 1, x_2, \cdots, x_n \rangle \\
r(\vec{x}) &= \langle x_2, x_3, \cdots, x_n, x_1 \rangle
\end{aligned}
$$

Namely, for every $i \in \{1, \cdots, n\}$,

$$
\sigma(\vec{x})(i) = \begin{cases} \vec{x}(1) - 1 & \text{if } i = 1 \\ \vec{x}(i) & \text{otherwise} \end{cases}
$$

and

$$
r(\vec{x})(i) = \begin{cases} \vec{x}(i+1) & \text{if } i < n \\ \vec{x}(1) & \text{if } i = n \end{cases}
$$

In particular, for every $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, n\}$

$$
r^i(\vec{x})(j) = \begin{cases} \vec{x}(j+i) & \text{if } j + i \leq n \\ \vec{x}(j+i-n) & \text{if } j + i > n \end{cases}
$$

By using this notation, the n-dimensional tarai function t can be defined as for every $\vec{x} \in \mathbb{Z}^n$, if $\vec{x}(1) \leq \vec{x}(2)$, then $t(\vec{x}) = \vec{x}(2)$, and if $\vec{x}(1) > \vec{x}(2)$, then $t(\vec{x}) = t(\vec{y})$, where $\vec{y} \in \mathbb{Z}^n$ is defined by $\vec{y}(i) = t(\sigma(r^{i-1}(\vec{x})))$ for all $i = 1, 2, \ldots, n$.

Let $k \in \mathbb{N}$ with $2 \leq k \leq n$. Define X_k to be the set of all $\vec{x} \in \mathbb{Z}^n$ such that for every $i < k$, $\vec{x}(i) \leq \vec{x}(k)$. Note that the family $\{X_k : 2 \leq k \leq n\}$ is not pairwise disjoint. For example, $\langle 2, 1, 4, 3, 5 \rangle \in X_3 \cap X_5$. Note also that if $\vec{x} \in \mathbb{Z}^n$ satisfies $\vec{x}(1) < \max \vec{x}$, then there exists a k such that $\vec{x} \in X_k$.

Lemma 1.1. *For every $k \in \mathbb{N}$ with $2 \leq k \leq n$, the following statement holds.*

$(*)_k$: *For every $\vec{x} \in X_k$,*
(1) *$t(\vec{x})$ terminates with call-by-need,*
(2) *$t(\vec{x})$ depends only on $\vec{x} \upharpoonright \{1, \cdots, k\}$, and*
(3) *$t(\vec{x}) \leq \vec{x}(k)$.*

Proof. Go by induction on k.

First assume $k = 2$. Then, we have $\vec{x}(1) \leq \vec{x}(2)$ and hence $t(\vec{x}) = \vec{x}(2)$. Hence, $\vec{x}$ clearly satisfies (i)–(iii).

Suppose that $(*)_{k'}$ holds for all $k' \in \mathbb{N}$ with $2 \leq k' < k$. We shall prove $(*)_k$. By way of contradiction, suppose that there exists an $\vec{x} \in X_k$ which does not satisfy one of (i)–(iii). By inductive hypothesis, we have $\vec{x} \notin X_{k-1}$. In particular, $\vec{x}(1) > \vec{x}(2)$. Hence, we can pick the least $y_1 \in \mathbb{Z}$ such that if $\vec{y}$ is defined as $\vec{y}(1) = y_1$ and $\vec{y}(i) = \vec{x}(i)$ for every $i \in \{2, \cdots, n\}$, then $\vec{y}$ does not satisfy one of (i)–(iii). By redefining $\vec{x}$, we may assume that for every $\vec{x}' \in \mathbb{N}^n$, if $\vec{x}'(1) < \vec{x}(1)$ and $\vec{x}'(i) = \vec{x}(i)$ for every $i \in \{2, \cdots, n\}$, then $\vec{x}'$ satisfies (i)–(iii).

For each $i \in \{1, \ldots, k\}$, define $\vec{z}_i = \sigma(r^{i-1}(\vec{y}))$ and $\vec{y}(i) = t(\vec{z}_i)$. For $i \in \{k+1, \ldots, n\}$, let $\vec{y}(i)$ be left undefined. By definition, $t(\vec{x})$ terminates with call-by-need if $t(\vec{z}_i)$ for all $i \in \{1, \ldots, k\}$ and $t(\vec{y})$ terminate with call-by-need, and in that case, $t(\vec{x}) = t(\vec{y})$.

Notice that for every $i \in \{1, \cdots, k-1\}$ and $j \in \{2, \cdots, k-i+1\}$,

$$\vec{z}_i(j) = \vec{x}(j+i-1) \leq \vec{x}(k)$$
$$\vec{z}_i(1) = \vec{x}(i) - 1 \leq \vec{x}(k)$$

and

$$\vec{z}_i(k-i+1) = \vec{x}(i+k-i+1-1) = \vec{x}(k)$$

Hence, $\vec{z}_i(k-i+1) = \vec{x}(k)$. Therefore, $\vec{z}_i \in X_{k-i+1}$. If $i \geq 2$, then by inductive hypothesis, $(*)_{k-i+1}$ holds. Thus, $\vec{z}_i$ satisfies (i)–(iii). In particular, $t(\vec{z}_i) \leq \vec{z}_i(k-i+1) = \vec{x}(k)$. If $i = 1$, we have $\vec{z}_1 = \sigma(\vec{x})$ and by the minimality of $\vec{x}(1)$, we know that $\vec{z}_1$ satisfies (i)–(iii). For every $i \in \{1, \cdots, k-1\}$, $\vec{y}(i) = t(\vec{z}_i) \leq \vec{x}(k)$. Moreover, $\vec{z}_{k-1}(2) = \vec{x}(k)$. Thus, $\vec{z}_{k-1}(1) \leq \vec{x}(k) = \vec{z}_{k-1}(2)$, which implies $\vec{y}(k-1) = t(\vec{z}_{k-1}) = \vec{z}_{k-1}(2) = \vec{x}(k)$. Hence, $\vec{y} \in X_{k-1}$ and so $\vec{y}$ satisfies (i)–(iii). It is now easy to see that $\vec{x}$ also satisfies (i)–(iii). $\square$(Lemma 1.1)

Theorem 1.2. *For every $\vec{x} \in \mathbb{Z}^n$, $t(\vec{x})$ terminates with call-by-need and $t(\vec{x}) \leq \max(\vec{x})$.*

Proof. By Lemma 1.1, it suffices to show that for every $\vec{x} \in \mathbb{Z}^n$ with $\vec{x}(1) = \max(\vec{x})$, $t(\vec{x})$ terminates with call-by-need.

By way of contradiction, suppose that $t(\vec{x})$ does not terminate. We may also assume that for every $\vec{x}' \in \mathbb{Z}^n$ with $\vec{x}'(1) < \vec{x}(1)$ and $\vec{x}'(i) = \vec{x}(i)$ for every $i \in \{2, \cdots, n\}$, $t(\vec{x}')$ terminates with call-by-need.

For every $i \in \{1, \cdots, n\}$, let $\vec{z}_i = \sigma(r^{i-1}(\vec{x}))$ and $\vec{y}(i) = t(\vec{z}_i)$. First suppose $i \geq 2$. Then, for every $j \in \{2, \cdots, n-i+1\}$,

$$\vec{z}_i(j) = \vec{x}(j+i-1) \leq \vec{x}(1)$$

and

$$\vec{z}_i(1) = \vec{x}(i) - 1 \leq \vec{x}(1)$$

Moreover, $\vec{z}_i(n-i+2) = \vec{x}(n-i+2+(i-1)-n) = \vec{x}(1)$. Thus, $\vec{z}_i \in X_{n-i+2}$. Hence, $t(\vec{z}_i)$ terminates with call-by-need and $t(\vec{z}_i) \leq \vec{x}(1)$. In addition, $\vec{z}_n(1) = \vec{x}(1+(n-1)) = \vec{x}(n) \leq \vec{x}(1)$ and $\vec{z}_n(2) = \vec{x}(2+(n-1)-n) = \vec{x}(1)$. Thus, $t(\vec{z}_n) = \vec{x}(1)$.

Suppose that $i = 1$. Then, $\vec{z}_1 = \sigma(\vec{x})$. By the minimality of $\vec{x}(1)$, $t(\vec{z}_1)$ terminates with call-by-need and $t(\vec{z}_1) \leq \vec{x}(1)$.

Therefore, for every $i \in \{1, \cdots, n\}$, we have $\vec{y}(i) = t(\vec{z}_i) \leq \vec{x}(1)$ and $t(\vec{z}_n) = \vec{x}(1)$. Hence, we have $\vec{y} \in X_n$. By $(*)_n$, $t(\vec{y})$ terminates with call-by-need and $t(\vec{y}) \leq \vec{y}(n) = \vec{x}(1)$. It follows that $t(\vec{x})$ also terminates with call-by-need and $t(\vec{x}) \leq \vec{x}(1)$. This contradicts the choice of $\vec{x}$. $\square$(Theorem 1.2)

2. Alternative definition of the function of T. Bailey and J. Cowles

In this section, we shall set up some definitions and notation which help us prove the main theorem. Let F denote the set of all non-empty finite sequences of integers. f denotes the function defined by T. Bailey and J. Cowles.

Let k be a function with domain F as follows. Let $\vec{x} \in F$ be of length n. If $\vec{x}(1) > \vec{x}(2) > \ldots > \vec{x}(n)$, then let $k(\vec{x}) = n$. Otherwise, let $k(\vec{x})$ be the least k such that $\vec{x}(k) \leq \vec{x}(k+1)$.

Let l be a function with domain F as follows. Let $\vec{x} \in F$ be of length n. If there is an integer l with $1 \leq l < k(\vec{x})$ such that $\vec{x}(l) > \vec{x}(l+1)+1$ and $\vec{x}(l+1) = \vec{x}(l+2)+1$, then let $l(\vec{x})$ be the least such l. Otherwise, let $l(\vec{x}) = k(\vec{x}) - 1$. Notice that $l(\vec{x}) = 0$ if and only if $k(\vec{x}) = 1$.

Lemma 2.1. *Let $\vec{x} \in F$ be of length $n \geq 3$. Suppose that $k(\vec{x}) = n-1$. Then, $g_b(\vec{x}) = \max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1))$.*

Proof. We shall prove the lemma by induction on n. If $n = 3$, then $g_b(\vec{x}) = \vec{x}(3)$. We also have $k(\vec{x}) = 2$ and $l(\vec{x}) = 1$. Thus, $\max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1)) = \vec{x}(3)$. Therefore, $g_b(\vec{x}) = \max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1))$.

Suppose that the conclusion holds for all $\vec{x}$ of length n for some $n \geq 3$. Let $\vec{x} \in F$ be of length $n+1$ with $k(\vec{x}) = n$. First suppose that $\vec{x}(1) > \vec{x}(2)+1$ and $\vec{x}(2) = \vec{x}(3)+1$. Then $g_b(\vec{x}) = \max(\vec{x}(3), \vec{x}(n+1))$. It is clear that $l(\vec{x}) = 1$. Thus, $\max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1)) = \max(\vec{x}(3), \vec{x}(n+1)) = g_b(\vec{x})$.

Suppose $\vec{x}(1) = \vec{x}(2)+1$ or $\vec{x}(2) > \vec{x}(3)+1$. Then $g_b(\vec{x}) = g_b(\vec{y})$ where $\vec{y}$ is a sequence of length n such that $\vec{y}(i) = \vec{x}(i+1)$ for every $i = 1, \ldots, n$.

Notice that $k(\vec{y}) = k(\vec{x}) - 1$ and $l(\vec{y}) = l(\vec{x}) - 1$. By inductive hypothesis, $g_b(\vec{y}) = \max(\vec{y}(l(\vec{y}) + 2), \vec{y}(k(\vec{y}) + 1)) = \max(\vec{y}(l(\vec{x}) + 1), \vec{y}(k(\vec{x}))) = \max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1))$. Therefore, $g_b(\vec{x}) = \max(\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1))$ □(Lemma 2.1)

By using the previous lemma, the following is immediate.

Lemma 2.2. *Let $\vec{x} \in F$ be of length $n \geq 3$. If $k(\vec{x}) = n$ (i.e. $\vec{x}(1) > \vec{x}(2) > \ldots > \vec{x}(n)$), then $f(\vec{x}) = \vec{x}(1)$. If $k(\vec{x}) < n$, then $f(\vec{x}) = \max(\vec{x}(l(\vec{x}) + 2), \vec{x}(k(\vec{x}) + 1))$.*

We shall use this characterization in the next section.

3. Closed form

In this section, we shall give a proof of the theorem of T. Bailey and J. Cowles. Note that if carefully rewritten, the proof simultaneously gives the termination of the n-dimensional tarai function. We chose to give separate proofs to simplify the arguments. Throughout this section, we fix a natural number n with $n \geq 3$.

We shall show that for every $\vec{x} \in \mathbb{Z}^n$, $t(\vec{x}) = f(\vec{x})$. To this end, by Theorem 1.2, it suffices to show that f satisfies the n-dimensional tarai recurrence.

We shall begin with some easy facts about f.

Lemma 3.1. *For every $\vec{x} \in \mathbb{Z}^n$, if $k(\vec{x}) \leq 2$, then $f(\vec{x}) = \vec{x}(k(\vec{x}) + 1)$.*

Proof. Since $k(\vec{x}) \leq 2$, we have $l(\vec{x}) = k(\vec{x}) - 1$. Thus, $f(\vec{x}) = \max\{\vec{x}(l(\vec{x}) + 2), \vec{x}(k(\vec{x}) + 1)\} = \vec{x}(k(\vec{x}) + 1)$. □(Lemma 3.1)

Lemma 3.2. *For every $\vec{x} \in \mathbb{Z}^n$ and $m \in \{1, \ldots, l(\vec{x}) + 2\}$, if $k(\vec{x}) < n$, and $\vec{x}(m) \leq \vec{x}(k(\vec{x}) + 1)$, then $f(\vec{x}) = \vec{x}(k(\vec{x}) + 1)$.*

Proof. If $l(\vec{x}) = k(\vec{x}) - 1$, then clearly $f(\vec{x}) = \max\{\vec{x}(l(\vec{x}) + 2), \vec{x}(k(\vec{x}) + 1)\} = \vec{x}(k(\vec{x}) + 1)$. Suppose $l(\vec{x}) < k(\vec{x}) - 1$. Then $l(\vec{x}) + 2 \leq k(\vec{x})$. We have $m \leq l(\vec{x}) + 2 \leq k(\vec{x})$. Thus,

$$\vec{x}(l(\vec{x}) + 2) \leq \vec{x}(m) \leq \vec{x}(k(\vec{x}) + 1)$$

So, $f(\vec{x}) = \max\{\vec{x}(l(\vec{x}) + 2), \vec{x}(k(\vec{x}) + 1)\} = \vec{x}(k(\vec{x}) + 1)$. □(Lemma 3.2)

Lemma 3.3. *For every $\vec{x} \in \mathbb{Z}^n$, if $k(\vec{x}) < n$ and $\vec{x}(3) \leq \vec{x}(k(\vec{x}) + 1)$, then $f(\vec{x}) = \vec{x}(k(\vec{x}) + 1)$.*

Proof. By Lemma 3.1, if $k(\vec{x}) \leq 2$, then $f(\vec{x}) = \vec{x}(k(\vec{x}) + 1)$. Suppose $k(\vec{x}) \geq 3$. Then by applying Lemma 3.2 with $m = 3$, we have $f(\vec{x}) = \vec{x}(k(\vec{x}) + 1)$. □(Lemma 3.3)

Lemma 3.4. *For every $\vec{x} \in \mathbb{Z}^n$, if $k(\vec{x}) < n$ and $\vec{x}(2) \leq \vec{x}(k(\vec{x})+1)$, then $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$.*

Proof. If $k(\vec{x}) \geq 3$, then we have $\vec{x}(3) < \vec{x}(2) \leq \vec{x}(k(\vec{x})+1)$. By Lemma 3.3, $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$. If $k(\vec{x}) \leq 2$, then by Lemma 3.1, we have $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$. □(Lemma 3.4)

Lemma 3.5. *For every $\vec{x} \in \mathbb{Z}^n$, if $k(\vec{x}) < n$ and $l(\vec{x}) \geq k(\vec{x}) - 2$, then $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$.*

Proof. By definition, $l(\vec{x}) \leq k(\vec{x}) - 1$. So, $l(\vec{x}) \geq k(\vec{x}) - 2$ implies either $l(\vec{x}) = k(\vec{x}) - 2$ or $l(\vec{x}) = k(\vec{x}) - 1$. If $l(\vec{x}) = k(\vec{x}) - 1$, then clearly $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$. Suppose $l(\vec{x}) = k(\vec{x}) - 2$. Then,

$$\begin{aligned} f(\vec{x}) &= \max\{\vec{x}(l(\vec{x})+2), \vec{x}(k(\vec{x})+1)\} \\ &= \max\{\vec{x}(k(\vec{x})), \vec{x}(k(\vec{x})+1)\} \\ &= \vec{x}(k(\vec{x})+1) \end{aligned}$$

since by the definition of $k(\vec{x})$, $\vec{x}(k(\vec{x})) \leq \vec{x}(k(\vec{x})+1)$. □(Lemma 3.5)

Lemma 3.6. *For every $\vec{x} \in \mathbb{Z}^n$, if $\vec{x}(2) \leq \vec{x}(3)$, then $f(\vec{x})$ is either $\vec{x}(2)$ or $\vec{x}(3)$. In particular, if $\vec{x}(2) = \vec{x}(3)$, then $f(\vec{x}) = \vec{x}(2)$.*

Proof. Since $\vec{x}(2) \leq \vec{x}(3)$, we have $k(\vec{x}) \leq 2$. Then, $k(\vec{x}) - 2 \leq 0 \leq l(\vec{x})$. By Lemma 3.5, $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$. Since $k(\vec{x})$ is either 1 or 2, $f(\vec{x})$ is either $\vec{x}(2)$ or $\vec{x}(3)$. □(Lemma 3.6)

Lemma 3.7. *For every $\vec{x} \in \mathbb{Z}^n$ and $m \in \{1, \dots, n-1\}$, if $\vec{x}(m) = \vec{x}(m+1)$, $k(\vec{x}) \geq m - 1$, and $l(\vec{x}) \geq m - 2$, then $f(\vec{x}) = \vec{x}(m)$.*

Proof. Since $\vec{x}(m) = \vec{x}(m+1)$, we have $k(\vec{x}) \leq m$. So,

$$l(\vec{x}) \geq m - 2 \geq k(\vec{x}) - 2$$

Hence, by Lemma 3.5, $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$. However, since $m-1 \leq k(\vec{x}) \leq m$, $k(\vec{x})$ is either $m-1$ or m. If $k(\vec{x}) = m-1$, then $\vec{x}(k(\vec{x})+1) = \vec{x}(m)$. If $k(\vec{x}) = m$, then $\vec{x}(k(\vec{x})+1) = \vec{x}(m+1) = \vec{x}(m)$ by assumption. □(Lemma 3.7)

Lemma 3.8. *For every $\vec{x} \in \mathbb{Z}^n$ and $m \in \{1, \dots, n-2\}$, if $k(\vec{x}) \geq m - 1$, $l(\vec{x}) \geq m - 2$, $\vec{x}(m) = \vec{x}(m+2)$, and $\vec{x}(m) \geq \vec{x}(m+1)$, then $f(\vec{x}) = \vec{x}(m)$.*

Proof. By assumption, $\vec{x}(m+1) \leq \vec{x}(m) = \vec{x}(m+2)$. So, $k(\vec{x}) \leq m+1$. Therefore, $k(\vec{x})$ is either $m-1$, m, or $m+1$.

Case 1. $k(\vec{x}) = m - 1$.

Then, $l(\vec{x}) \geq m-2 = k(\vec{x})-1$. By Lemma 3.5, $f(\vec{x}) = \vec{x}(k(\vec{x})+1) = \vec{x}(m)$.

Case 2. $k(\vec{x}) = m$.

In this case, we have $\vec{x}(m) \leq \vec{x}(m+1)$. By assumption, we also have $\vec{x}(m+1) \leq \vec{x}(m)$. Therefore, $\vec{x}(m) = \vec{x}(m+1)$. By Lemma 3.7, $f(\vec{x}) = \vec{x}(m)$.

Case 3. $k(\vec{x}) = m+1$ and $l(\vec{x}) = m-2$.

Then, $\vec{x}(k(\vec{x})+1) = \vec{x}(m+2) = \vec{x}(m)$ and $\vec{x}(l(\vec{x})+2) = \vec{x}(m)$. Thus, $f(\vec{x}) = \vec{x}(m)$.

Case 4. $k(\vec{x}) = m+1$ and $l(\vec{x}) \geq m-1$.

Note that $l(\vec{x}) \geq m-1 = k(\vec{x})-2$. By Lemma 3.5, $f(\vec{x}) = \vec{x}(k(\vec{x})+1) = \vec{x}(m+2) = \vec{x}(m)$. □(Lemma 3.8)

Lemma 3.9. *For every $\vec{x} \in \mathbb{Z}^n$, if $2 \leq k(\vec{x}) < n$, then $f(\vec{x}) \leq \max\{\vec{x}(3), \vec{x}(k(\vec{x})+1)\}$.*

Proof. If $f(\vec{x}) = \vec{x}(k(\vec{x})+1)$, it is trivial. Thus, we assume that $f(\vec{x}) = \vec{x}(l(\vec{x})+2) \neq \vec{x}(k(\vec{x})+1)$. It is easy to see that $l(\vec{x}) \leq k(\vec{x})-2$. So, $l(\vec{x})+2 \leq k(\vec{x})$. If $l(\vec{x}) = 0$, then $k(\vec{x}) = 1$, which contradicts $l(\vec{x}) \leq k(\vec{x})-2$. So, we have $l(\vec{x}) \geq 1$. Therefore, $3 \leq l(\vec{x})+2 \leq k(\vec{x})$. Hence, $\vec{x}(3) \geq \vec{x}(l(\vec{x})+2) = f(\vec{x})$. □(Lemma 3.9)

Lemma 3.10. *For every $\vec{x} \in \mathbb{Z}^n$, if $k(\vec{x}) < n$, then $f(\vec{x})$ satisfies the tarai recurrence.*

Proof. Let $k = k(\vec{x})$ and $l = l(\vec{x})$. If $k = 1$, it is trivial. Assume $k \geq 2$.

For each $i = 1, \cdots, n$, define $\vec{z}_i = \sigma(r^{i-1}(\vec{x}))$ and let $\vec{y} \in \mathbb{Z}^n$ be defined by $\vec{y}(i) = f(\vec{z}_i)$. It suffices to show $f(\vec{x}) = f(\vec{y})$.

It is easy to see that

$$\vec{z}_i(j) = \begin{cases} \vec{x}(i)-1 & \text{if } j = 1 \\ \vec{x}(j+i-1) & \text{if } j+i-1 \leq n \\ \vec{x}(j+i-1-n) & \text{if } j+i-1 > n \end{cases}$$

Define m to be the least such that $\vec{x}(m) > \vec{x}(m+1)+1$ or $m = k-1$. Clearly we have $m \leq l$.

Claim 1. $\vec{y}(k) = \vec{x}(k+1)$

⊢ Note that $\vec{z}_k(1) = \vec{x}(k)-1$ and $\vec{z}_k(2) = \vec{x}(k+1)$. Since $\vec{x}(k) \leq \vec{x}(k+1)$, we have $\vec{z}_k(1) \leq \vec{z}_k(2)$. Hence, $\vec{y}(k) = f(\vec{z}_k) = \vec{z}_k(2) = \vec{x}(k+1)$. ⊣ (Claim 1)

Claim 2. $k(\vec{y}) \leq k-1$.

⊢ Note $\vec{z}_{k-1}(2) = \vec{x}(k)$ and $\vec{z}_{k-1}(3) = \vec{x}(k+1)$. So, $\vec{z}_{k-1}(2) \leq \vec{z}_{k-1}(3)$. By Lemma 3.6, $\vec{y}(k-1)$ is either $\vec{z}_{k-1}(2) = \vec{x}(k)$ or $\vec{z}_{k-1}(3) = \vec{x}(k+1)$. In either way, we have $\vec{y}(k-1) \leq \vec{x}(k+1) = \vec{y}(k)$. Thus, $k(\vec{y}) \leq k-1$. ⊣ (Claim 2)

Claim 3. For every $i \in \{1, \ldots, n-1\}$, if $\vec{x}(i) = \vec{x}(i+1)+1$, then $\vec{y}(i) = \vec{x}(i+1)$. In particular, if $i < m$, then $\vec{y}(i) = \vec{x}(i+1)$.

$\vdash$ We have $\vec{z}_i(1) = \vec{x}(i) - 1$ and $\vec{z}_i(2) = \vec{x}(i+1)$. Thus, $\vec{z}_i(1) = \vec{z}_i(2)$. So, $\vec{y}(i) = \vec{z}_i(2) = \vec{x}(i+1)$. $\dashv$ (Claim 3)

Claim 4. For every $i \in \{1, \ldots, m-2\}$, $\vec{y}(i) = \vec{y}(i+1)+1$. In particular, $k(\vec{y}) \geq m-1$ and $l(\vec{y}) \geq m-2$. If $l(\vec{y}) < k(\vec{y}) - 1$, then $l(\vec{y}) \geq m-1$.

$\vdash$ If $i \in \{1, \ldots, m-2\}$, then by Claim 3, $\vec{y}(i) = \vec{x}(i+1)$ and $\vec{y}(i+1) = \vec{x}(i+2)$. Since $i+1 < m$, we have $\vec{x}(i+1) = \vec{x}(i+2)+1$. Thus, $\vec{y}(i) = \vec{y}(i+1)+1$. So we have $k(\vec{y}) \geq m-1$. If $l(\vec{y}) = k(\vec{y}) - 1$, then $l(\vec{y}) \geq m-1-1 = m-2$. If $l(\vec{y}) < k(\vec{y}) - 1$, then we have $\vec{y}(l(\vec{y})) > \vec{y}(l(\vec{y})+1)+1$. But we also have $\vec{y}(m-2) = \vec{y}(m-1)$ and hence $l(\vec{y}) \neq m-2$. So, $l(\vec{y}) \geq m-1$. $\dashv$ (Claim 4)

Claim 5. For every $i \in \{1, \ldots, k-1\}$, if $\vec{x}(i) > \vec{x}(i+1)+1$, then $k(\vec{z}_i) = k-i+1$ and $\vec{z}_i(k(\vec{z}_i)+1) = \vec{x}(k+1)$.

$\vdash$ Note $\vec{z}_i(1) = \vec{x}(i) - 1$ and $\vec{z}_i(2) = \vec{x}(i+1)$. By assumption, $\vec{z}_i(1) > \vec{z}_i(2)$. For every $j \in \{2, \ldots, k-i\}$, we have $\vec{z}_i(j) = \vec{x}(i+j-1)$ and $\vec{z}_i(j+1) = \vec{x}(i+j)$. Since $i+j \leq i+k-i = k$, we have $\vec{x}(i+j-1) > \vec{x}(i+j)$ and hence $\vec{z}_i(j) > \vec{z}_i(j+1)$. Thus, $k(\vec{z}_i) \geq k-i+1$. Note $\vec{z}_i(k-i+1) = \vec{x}(i+(k-i+1)-1) = \vec{x}(k)$ and $\vec{z}_i(k-i+2) = \vec{x}(i+(k-i+2)-1) = \vec{x}(k+1)$. By definition, $\vec{x}(k) \leq \vec{x}(k+1)$ and hence $\vec{z}_i(k-i+1) \leq \vec{z}_i(k-i+2)$ Therefore, $k(\vec{z}_i) = k-i+1$. As we have already seen, $\vec{z}_i(k(\vec{z}_i)+1) = \vec{z}_i(k-i+2) = \vec{x}(k+1)$. $\dashv$ (Claim 5)

Claim 6. For every $i \in \{1, \ldots, l-1\}$, if $\vec{x}(i) > \vec{x}(i+1)+1$, then $l(\vec{z}_i) = l-i+1$ and hence $\vec{z}_i(l(\vec{z}_i)+2) = \vec{x}(l+2)$.

$\vdash$

Since $\vec{x}(i) > \vec{x}(i+1)+1$, we have $\vec{z}_i(1) > \vec{z}_i(2)$. Since $i < l$ and $\vec{x}(i) > \vec{x}(i+1)+1$, we have $\vec{z}_i(2) = \vec{x}(i+1) > \vec{x}(i+2)+1 = \vec{z}_i(3)+1$. Thus, $l(\vec{z}_i) \geq 2$.

Let $j \in \{2, \ldots, l-i\}$. Then, $\vec{z}_i(j) = \vec{x}(i+j-1)$, $\vec{z}_i(j+1) = \vec{x}(i+j)$, and $\vec{z}_i(j+2) = \vec{x}(i+j+1)$. Note $i+j-1 \leq i+(l-i)-1 = l-1$ and hence $i+j \leq l < k$. So, either $\vec{x}(i+j-1) = \vec{x}(i+j)+1$ or $\vec{x}(i+j) > \vec{x}(i+j+1)+1$. Thus, either $\vec{z}_i(j) = \vec{z}_i(j+1)+1$ or $\vec{z}_i(j+1) > \vec{z}_i(j+2)+1$. Hence $l(\vec{z}_i) \geq l-i+1$. If $l = k-1$, then by Claim 5, $k(\vec{z}_i) = k-i+1$ and hence $l(\vec{z}_i) \leq k(\vec{z}_i) - 1 = k-i = l-i+1$. Thus, $l(\vec{z}_i) = l-i+1$. If $l < k-1$, then we have both $\vec{x}(l) > \vec{x}(l+1)+1$ and $\vec{x}(l+1) = \vec{x}(l+2)+1$. Note $l-i+1 \geq l-(l-1)+1 = 2$. So, $\vec{z}_i(l-i+1) = \vec{x}(l)$, $\vec{z}_i(l-i+2) = \vec{x}(l+1)$ and $\vec{z}_i(l-i+3) = \vec{x}(l+2)$. Thus, $\vec{z}_i(l-i+1) > \vec{z}_i(l-i+2)+1$ and $\vec{z}_i(l-i+2) = \vec{z}_i(l-i+3)+1$. So, $l(\vec{z}_i) = l-i+1$. $\dashv$ (Claim 6)

Claim 7. For every $i \in \{1, \ldots, l-1\}$, if $\vec{x}(i) > \vec{x}(i+1)+1$, then $f(\vec{z}_i) = f(\vec{x})$.

$\vdash$ By Claim 5, $\vec{z}_i(k(\vec{z}_i) + 1) = \vec{x}(k + 1)$. By Claim 6, $\vec{z}_i(l(\vec{z}_i) + 2) = \vec{x}(l + 2)$. Therefore,

$$\begin{aligned} f(\vec{z}_i) &= \max\{\vec{z}_i(l(\vec{z}_i) + 2), \vec{z}_i(k(\vec{z}_i) + 1)\} \\ &= \max\{\vec{x}(l + 2), \vec{x}(k + 1)\} = f(\vec{x}) \end{aligned}$$

$\dashv$ (Claim 7)

Case 1. $m + 2 \leq l$.

Then, by Claim 7, $\vec{y}(m) = \vec{y}(m + 1) = f(\vec{x})$. By Claim 4, $k(\vec{y}) \geq m - 1$ and $l(\vec{y}) \geq m - 2$. By Lemma 3.7, $f(\vec{y}) = f(\vec{x})$.

Case 2. $m + 1 = l$.

Then by Claim 7, $\vec{y}(m) = f(\vec{x})$.

Subcase 2.1. $l + 1 = k$.

Then, we have $f(\vec{x}) = \vec{x}(k + 1)$ and

$$\vec{y}(m + 2) = \vec{y}(l + 1) = \vec{y}(k) = \vec{x}(k + 1) = f(\vec{x})$$

Consider $\vec{z}_l$. We have

$$\begin{aligned} \vec{z}_l(1) &= \vec{x}(l) - 1 \\ \vec{z}_l(2) &= \vec{x}(l + 1) = \vec{x}(k) \\ \vec{z}_l(3) &= \vec{x}(l + 2) = \vec{x}(k + 1) \end{aligned}$$

By the definition of k, $\vec{x}(k) \leq \vec{x}(k+1)$. Thus, $\vec{z}_l(2) \leq \vec{z}_l(3)$. By Lemma 3.6, $f(\vec{z}_l)$ is either $\vec{z}_l(2)$ or $\vec{z}_l(3)$, i.e. either $\vec{x}(k)$ or $\vec{x}(k + 1)$. In particular, we have $f(\vec{z}_l) \leq \vec{x}(k + 1)$ and hence $\vec{y}(m + 1) = \vec{y}(m) \leq \vec{x}(k + 1)$. Therefore, we have $k(\vec{y}) \geq m - 1$, $l(\vec{y}) \geq m - 2$, $\vec{y}(m) = \vec{y}(m + 2) = \vec{x}(k + 1)$, and $\vec{y}(m + 1) \leq \vec{y}(m)$. By Lemma 3.8, we have $f(\vec{y}) = \vec{x}(k + 1) = f(\vec{x})$.

Subcase 2.2. $l < k - 1$.

Then, we have $\vec{x}(l) > \vec{x}(l + 1) + 1$. By Claim 5, $k(\vec{z}_l) = k - l + 1$ and $\vec{z}_l(k(\vec{z}_l) + 1) = \vec{x}(k + 1)$.

Subsubcase 2.2.1. $\vec{x}(l + 2) \leq \vec{x}(k + 1)$

Then, $\vec{z}_l(3) = \vec{x}(l + 3 - 1) = \vec{x}(l + 2) \leq \vec{x}(k + 1)$ By Lemma 3.3, $f(\vec{z}_l) = \vec{z}_l(k(\vec{z}_l) + 1) = \vec{x}(k + 1)$. Therefore, we have $\vec{y}(m) = \vec{y}(m + 1) = \vec{x}(k + 1)$. Since $k(\vec{y}) \geq m-1$ and $l(\vec{y}) \geq m-2$, by Lemma 3.7, we have $f(\vec{y}) = \vec{y}(m) = \vec{x}(k + 1) = f(\vec{x})$.

Subsubcase 2.2.2. $\vec{x}(l + 2) > \vec{x}(k + 2)$

Recall $k(\vec{z}_l) = k - l + 1$. Note $k - l + 1 \geq 1 + 1 = 2$. By Lemma 3.9,

$$\begin{aligned} f(\vec{z}_l) &\leq \max\{\vec{z}_l(3), \vec{z}_l(k(\vec{z}) + 1)\} \\ &= \max\{\vec{x}(l + 2), \vec{x}(k + 2)\} = \vec{x}(l + 2) \end{aligned}$$

Therefore, $\vec{y}(m + 1) = \vec{y}(l) \leq \vec{x}(l + 2)$. Since $l < k - 1$, we have $\vec{x}(l + 1) = \vec{x}(l + 2) + 1$. So, $\vec{z}_{l+1}(1) = \vec{x}(l + 1) - 1 = \vec{x}(l + 2) = \vec{z}_{l+1}(2)$. Therefore, $f(\vec{z}_{l+1}) = \vec{z}_{l+1}(2) = \vec{x}(l + 2)$. Hence,

$$\begin{aligned} \vec{y}(m) &= \vec{x}(l + 2) \\ \vec{y}(m + 1) &\leq \vec{x}(l + 2) \\ \vec{y}(m + 2) &= \vec{x}(l + 2) \end{aligned}$$

By Lemma 3.8, we have $f(\vec{y}) = \vec{y}(m) = \vec{x}(l + 2) = f(\vec{x})$.

Case 3. $m = l$

Subcase 3.1. $l = k - 1$.

Claim 8. $\vec{y}(l) = f(\vec{z}_l)$ is either $\vec{x}(k)$ or $\vec{x}(k + 1)$. In particular, $\vec{y}(l) = f(\vec{z}_l) \leq \vec{x}(k + 1) = f(\vec{x})$.

$\vdash$ Since we assumed $l = k - 1$, $\vec{y}(l + 1) = \vec{y}(k) = \vec{x}(k + 1)$. Note

$$\begin{aligned} \vec{z}_l(2) &= \vec{x}(l + 1) = \vec{x}(k) \\ &\leq \vec{x}(k + 1) = \vec{x}(l + 2) = \vec{z}_l(3) \end{aligned}$$

Thus, by Lemma 3.6, $f(\vec{z}_l)$ is either $\vec{x}(k)$ or $\vec{x}(k + 1)$. $\dashv$ (Claim 8)

Claim 9. $k(\vec{y}) \leq l$.

$\vdash$ Because

$$\vec{y}(l) \leq \vec{x}(k + 1) = \vec{y}(k) = \vec{y}(l + 1)$$

$\dashv$ (Claim 9)

Subsubcase 3.1.1. $l = 1$

Then, by Claim 8, $\vec{y}(1) = \vec{y}(l) \leq \vec{x}(k + 1)$. By Claim 1, $\vec{y}(2) = \vec{x}(k) = \vec{x}(k + 1)$. So, $f(\vec{y}) = \vec{x}(k + 1) = f(\vec{x})$.

Subsubcase 3.1.2. $l \geq 2$ and $\vec{x}(l) \leq \vec{x}(k + 1)$.

Recall that by Claim 8, $\vec{y}(l)$ is either $\vec{x}(k + 1)$ or $\vec{x}(k)$. If $\vec{y}(l) = \vec{x}(k + 1)$, then since $\vec{y}(l - 1) = \vec{x}(l) \leq \vec{x}(k + 1) = \vec{y}(l)$, we have $k(\vec{y}) \leq l - 1$. Since we also know $k(\vec{y}) \geq m - 1 = l - 1$ by Claim 4, we have $k(\vec{y}) = l - 1$. Since $l - 2 = m - 2 \leq l(\vec{y}) \leq k(\vec{y}) - 1 = l - 2$, we have $l(\vec{y}) = l - 2$. Therefore, $f(\vec{y}) = \vec{x}(k + 1) = f(\vec{x})$.

Suppose $\vec{y}(l) = \vec{x}(k)$. Since $l < k$, we have $\vec{y}(l - 1) = \vec{x}(l) > \vec{x}(k) = \vec{y}(l)$. Thus, $k(\vec{y}) \geq l$. By Claim 9, $k(\vec{y}) \leq l$ and hence $k(\vec{y}) = l$. Note that $l(\vec{y}) \geq m - 2 = l - 2 = k(\vec{y}) - 2$. By Lemma 3.5, $f(\vec{y}) = \vec{y}(k(\vec{y}) + 1) = \vec{y}(l + 1) = \vec{x}(k + 1) = f(\vec{x})$. Therefore, in either case, we get $f(\vec{y}) = f(\vec{x})$.

Subsubcase 3.1.3. $l \geq 2$ and $\vec{x}(l) > \vec{x}(k+1)$.

We have $\vec{y}(l-1) = \vec{x}(l) > \vec{x}(k+1) = \vec{y}(l)$. Hence, $k(\vec{y}) \geq l$. By Claim 9, we have $k(\vec{y}) = l$. Note that $l(\vec{y}) \geq m-2 = l-2 = k(\vec{y})-2$. By Lemma 3.5, we have $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = \vec{y}(l+1) = \vec{x}(k+1) = f(\vec{x})$.

Subcase 3.2. $l < k-1$.

Claim 10. $\vec{y}(l+1) = \vec{x}(l+2)$

$\vdash$ Since $l < k-1$, we have $\vec{x}(l) > \vec{x}(l+1)+1$ and $\vec{x}(l+1) = \vec{x}(l+2)+1$. Thus, $\vec{y}(l+1) = \vec{x}(l+2)$. $\dashv$ (Claim 10)

Subsubcase 3.2.1. $\vec{x}(l+2) \geq \vec{x}(k+1)$

By Claim 5, $k(\vec{z}_l) = k-l+1$ and $\vec{z}_l(k(\vec{z}_l)+1) = \vec{x}(k+1)$. We have

$$\begin{aligned} l &< k-1 \\ 1 &< k-l \\ 2 &< k-l+1 \end{aligned}$$

By Lemma 3.9,

$$\begin{aligned} \vec{y}(l) = f(\vec{z}_l) &\leq \max\{\vec{z}_l(3), \vec{z}_l(k(\vec{z}_l)+1)\} \\ &= \max\{\vec{x}(l+2), \vec{x}(k+1)\} = \vec{x}(l+2) = f(\vec{x}) \end{aligned}$$

Then, $\vec{y}(l) \leq \vec{x}(l+2) = \vec{y}(l+1)$. So, we have $k(\vec{y}) \leq l$. If $l = 1$, then clearly $k(\vec{y}) = 1 = l$. If $l \geq 2$, then since $l < l+2 \leq k$,

$$\vec{y}(l-1) = \vec{x}(l) > \vec{x}(l+2) \geq \vec{y}(l)$$

So, $k(\vec{y}) \geq l$ and hence $k(\vec{y}) = l$. Therefore, in either case, we get $k(\vec{y}) = l$.

We also have $l(\vec{y}) \geq m-2 = l-2 = k(\vec{y})-2$. By Lemma 3.5, $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = \vec{y}(l+1) = \vec{x}(l+2) = f(\vec{x})$.

Now we concentrate on the case $\vec{x}(l+2) < \vec{x}(k+1)$.

Claim 11. If $\vec{x}(l+2) < \vec{x}(k+1)$, then $\vec{y}(l) = \vec{x}(k+1) = f(\vec{x})$.

$\vdash$ Since $l < k-1$, we have $\vec{x}(l) > \vec{x}(l+1)+1$. By Claim 5, $\vec{z}_l(k(\vec{z}_l)+1) = \vec{x}(k+1)$. In addition, $\vec{z}_l(3) = \vec{x}(l+2) < \vec{x}(k+1)$. By Lemma 3.3, $\vec{y}(l) = f(\vec{z}_l) = \vec{x}(k+1)$. $\dashv$ (Claim 11)

Subsubcase 3.2.2. $\vec{x}(l+2) < \vec{x}(k+1)$ and $k(\vec{y}) = l-1$.

Note $l(\vec{y}) \geq l-2 \geq k(\vec{y})-1$. By Lemma 3.2, $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = \vec{y}(l) = f(\vec{x})$.

Subsubcase 3.2.3. $\vec{x}(l+2) < \vec{x}(k+1)$ and $k(\vec{y}) \geq l$.

Let p be the least such that $l+1 \leq p < k$ and $\vec{x}(p) > \vec{x}(p+1)+1$ if exists. Otherwise, let $p = k$.

Claim 12. $p \geq l+2$.

$\vdash$ By definition, $p \geq l+1$. Since $l < k-1$, $\vec{x}(l+1) = \vec{x}(l+2)+1$. So, $p \geq l+2$. $\dashv$ (Claim 12)

Claim 13. $k(\vec{y}) = p-1$.

$\vdash$ By assumption, we have $k(\vec{y}) \geq l$.

Subclaim 13.1. $\vec{y}(l) > \vec{y}(l+1)$. In particular, $k(\vec{y}) \geq l+1$.

$\vdash$ By Claim 11, $\vec{y}(l) = \vec{x}(k+1)$. By assumption, $\vec{x}(k+1) > \vec{x}(l+2) = \vec{y}(l+1)$. $\dashv$ (Subclaim 13.1)

Subclaim 13.2. For every $i \in \{l+1, \ldots, p-2\}$, $\vec{y}(i) = \vec{y}(i+1)+1$. In particular, $k(\vec{y}) \geq p-1$.

$\vdash$ By the definition of p, since $l < i < i+1 < p$, $\vec{x}(i) = \vec{x}(i+1)+1$ and $\vec{x}(i+1) = \vec{x}(i+2)+1$. Thus, $\vec{y}(i) = \vec{x}(i+1)$ and $\vec{y}(i+1) = \vec{x}(i+2)$. So, $\vec{y}(i) = \vec{x}(i+1) = \vec{x}(i+2)+1 = \vec{y}(i+1)+1$. $\dashv$ (Subclaim 13.2)

Subclaim 13.3. $\vec{y}(p) = \vec{x}(k+1)$.

$\vdash$ If $p = k$, then we have $\vec{y}(p) = \vec{y}(k) = \vec{x}(k+1)$. If $p = k-1$, then we have $\vec{z}_p(1) = \vec{x}(p)-1 > \vec{x}(p+1) = \vec{z}_p(2)$ and $\vec{z}_p(2) = \vec{x}(p+1) = \vec{x}(k) \leq \vec{x}(k+1) = \vec{z}_p(3)$. So, $\vec{y}(p) = f(\vec{z}_p) = \vec{x}(k+1)$.

Suppose $p \leq k-2$. By Claim 5, $\vec{z}_p(k(\vec{z}_p)+1) = \vec{x}(k+1)$. Note $l+2 \leq p \leq p+2 \leq k$, so, $\vec{z}_p(3) = \vec{x}(p+2) \leq \vec{x}(l+2) < \vec{x}(k+1) = \vec{z}_p(k(\vec{z}_p)+1)$. By Lemma 3.3, $y(p) = f(\vec{z}_p) = \vec{z}_p(k(\vec{z}_p)+1) = \vec{x}(k+1)$. $\dashv$ (Subclaim 13.3)

Subclaim 13.4. $\vec{y}(p-1) < \vec{x}(k+1) = \vec{y}(p)$. In particular, $k(\vec{z}) \leq p-1$.

$\vdash$ Since $p \geq l+2$, we have $p-1 \geq l+1 > l$. By the definition of p, we have $\vec{x}(p-1) = \vec{x}(p)+1$ and hence $\vec{y}(p-1) = \vec{x}(p)$. Since $l+2 \leq p \leq k$, we have $\vec{x}(p) \leq \vec{x}(l+2)$. By assumption, $\vec{x}(l+2) < \vec{x}(k+1)$. Therefore, $\vec{y}(p-1) < \vec{x}(k+1)$. $\dashv$ (Subclaim 13.4)

By Subclaim 13.2 and Subclaim 13.4, we have $k(\vec{y}) = p-1$. $\dashv$ (Claim 13)

If $l(\vec{y}) = k(\vec{y})-1$, then clearly $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = \vec{y}(p) = \vec{x}(k+1) = f(\vec{x})$. Suppose $l(\vec{y}) < k(\vec{y})-1$. By Claim 4, $l(\vec{y}) \geq m-1 = l-1$. Thus, $l+1 \leq l(\vec{y})+2$. Recall $\vec{y}(l+1) = \vec{x}(l+2) < \vec{x}(k+1) = \vec{y}(p) = \vec{y}(k(\vec{y})+1)$. By Lemma 3.2, $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = f(\vec{x})$. $\square$(Lemma 3.10)

Lemma 3.11. *For every $\vec{x} \in \mathbb{Z}^n$, $f(\vec{x})$ satisfies the tarai recurrence.*

Proof. By Lemma 3.10, we may assume that $k(\vec{x}) = n$, i.e. $\vec{x}(1) > \vec{x}(2) > \cdots > \vec{x}(n)$. For every $i = 1, \cdots, n$, define $\vec{z}_i = \sigma(r^{i-1}(\vec{x}))$ and $\vec{y}(i) = f(\vec{z}_i)$. We need to show that $f(\vec{y}) = f(\vec{x}) = \vec{x}(1)$.

Claim 1. $\vec{y}(1) < \vec{x}(1)$.

$\vdash$ We have $\vec{z}_1(1) = \vec{x}(1) - 1$ and $\vec{z}_1(i) = \vec{x}(i)$ for every $i = 2, \ldots, n$. Then, clearly we have $\vec{y}(1) = f(\vec{z}_1) \leq \max \vec{z}_1 = \vec{x}(1) - 1$. $\dashv$ (Claim 1)

Claim 2. $\vec{y}(n) = \vec{x}(1)$

$\vdash$ Since $\vec{z}_n(1) = \vec{x}(n)$ and $\vec{z}_n(2) = \vec{x}(1)$, we have $\vec{y}(n) = f(\vec{z}_n) = \vec{z}(2) = \vec{x}(1)$. $\dashv$ (Claim 2)

Claim 3. For every $i = 2, \ldots, n-1$, either $\vec{y}(i) = \vec{x}(1)$ or $\vec{y}(i) = \vec{x}(i+1)$.

$\vdash$ Note

$$\begin{aligned}
\vec{z}_i(1) &= \vec{x}(i) - 1 \\
\vec{z}_i(j) &= \vec{x}(j + i - 1) \text{ (for all } j = 2, \ldots, n - i + 1) \\
\vec{z}_i(n - i + 2) &= \vec{x}(1)
\end{aligned}$$

If $\vec{x}(i) - 1 = \vec{x}(i+1)$, then we have $\vec{z}_i(1) = \vec{z}_i(2)$ and hence $\vec{y}(i) = f(\vec{z}_i) = \vec{z}_i(2) = \vec{x}(i+1)$.

Suppose $\vec{x}(i) - 1 > \vec{x}(i+1)$. Then, for every $j = 2, \ldots, n-i$, we have $\vec{z}_i(j) = \vec{x}(j+i-1) > \vec{x}(j+i) = \vec{z}_i(j+1)$. Moreover, $\vec{z}_i(n-i+1) = \vec{x}(n) < \vec{x}(1) = \vec{z}_i(n-i+2)$. So, $k(\vec{z}_i) = n-i+1$ and $\vec{z}_i(k(\vec{z}_i)+1) = \vec{z}_i(n-i+2) = \vec{x}(1)$. Then, since $\vec{z}_i(2) = \vec{x}(i+1) < \vec{x}(1) = \vec{z}_i(k(\vec{z}_i)+1)$, by Lemma 3.4, we have $\vec{y}(i) = f(\vec{z}_i) = \vec{z}_i(k(\vec{z}_i)+1) = \vec{x}(1)$. $\dashv$ (Claim 3)

Let m be the least such that $\vec{y}(m+1) = \vec{x}(1)$. Since $\vec{y}(n) = \vec{x}(1)$, there is such an $m \leq n-1$. If $m = 1$, then we have $\vec{y}(1) < \vec{x}(1) = \vec{y}(2)$ and hence $f(\vec{y}) = \vec{y}(2) = \vec{x}(1)$.

Suppose $m > 1$. Then for every $i = 2, \ldots, m$, by Claim 3, we have $\vec{y}(i) = \vec{x}(i+1)$. Hence, for every $i = 2, \ldots, m-1$, we have $\vec{y}(i) > \vec{y}(i+1)$. We also have $\vec{y}(m) = \vec{x}(m+1) < \vec{x}(1) = \vec{y}(m+1)$. Therefore, $k(\vec{y}) = m$ and $\vec{y}(m+1) = \vec{x}(1)$. In particular, $\vec{y}(2) = \vec{x}(3) < \vec{x}(1) = \vec{y}(m+1) = \vec{y}(k(\vec{y})+1)$. By Lemma 3.4, we have $f(\vec{y}) = \vec{y}(k(\vec{y})+1) = \vec{x}(1)$. $\square$(Lemma 3.11)

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